Engineering Mathematics (4320002) - Summer 2024 Solutions

Q.1 [14 marks]

Fill in the blanks using appropriate choice from the given options.

Q.1.1 [1 mark]

Order of the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 3 & 4 \end{bmatrix}$ **is __.**

Answer: (b) 3 × 2

Solution:
Order of a matrix is given by (number of rows) × (number of columns)
Matrix A has 3 rows and 2 columns
Therefore, order = 3 × 2

Q.1.2 [1 mark]

If $A = \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$ **then** $A^{-1} = \_$

Answer: (d) A^T

Solution:
For orthogonal matrices, $A^{-1} = A^T$
Since $AA^T = I$, we have $A^{-1} = A^T$

Q.1.3 [1 mark]

$\begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix} \times \begin{bmatrix} -1 & 6 \\ 2 & 1 \end{bmatrix} = \_$

Answer: (a) $\begin{bmatrix} 3 & 8 \\ -5 & 30 \end{bmatrix}$

Solution:
$\begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix} \times \begin{bmatrix} -1 & 6 \\ 2 & 1 \end{bmatrix}$
$= \begin{bmatrix} 1(-1)+2(2) & 1(6)+2(1) \\ 5(-1)+0(2) & 5(6)+0(1) \end{bmatrix}$
$= \begin{bmatrix} -1+4 & 6+2 \\ -5+0 & 30+0 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ -5 & 30 \end{bmatrix}$

Q.1.4 [1 mark]

If $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ **then** $A^T =$ _

Answer: (b) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Solution:
Transpose of a matrix is obtained by interchanging rows and columns
$A^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Q.1.5 [1 mark]

$\frac{d}{dx}(4^x) =$ _

Answer: (a) $4^x \log_e 4$

Solution:
$\frac{d}{dx}(a^x) = a^x \ln a$
Therefore, $\frac{d}{dx}(4^x) = 4^x \ln 4 = 4^x \log_e 4$

Q.1.6 [1 mark]

$\frac{d}{dx}(\sin^2 x + \cos^2 x) =$ _

Answer: (b) 0

Solution:
$\sin^2 x + \cos^2 x = 1$ (trigonometric identity)
$\frac{d}{dx}(1) = 0$

Q.1.7 [1 mark]

If $x = \sin\theta, y = \cos\theta$ **then** $\frac{dy}{dx} =$ _

Answer: (d) $-\cot\theta$

Solution:
$\frac{dx}{d\theta} = \cos\theta$, $\frac{dy}{d\theta} = -\sin\theta$
$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta = -\cot\theta$

Q.1.8 [1 mark]

$\int x^7 dx =$ _

Answer: (c) $\frac{x^8}{8}$

Solution:
$\int x^n dx = \frac{x^{n+1}}{n+1} + c$
$\int x^7 dx = \frac{x^8}{8} + c$

Q.1.9 [1 mark]

$\int_{-2}^{2} x^5 dx =$ _

Answer: (b) 0

Solution:
x^5 is an odd function
For odd functions, $\int_{-a}^{a} f(x)dx = 0$
Therefore, $\int_{-2}^{2} x^5 dx = 0$

Q.1.10 [1 mark]

$\int \frac{\cos x}{\sin x} dx =$ _

Answer: (d) $\log|\sin x|$

Solution:
Let $u = \sin x$, then $du = \cos x dx$
$\int \frac{\cos x}{\sin x} dx = \int \frac{du}{u} = \log|u| + c = \log|\sin x| + c$

Q.1.11 [1 mark]

The order of the differential equation $\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^4 + y = 0$ is _

Answer: (a) 3

Solution:
Order of a differential equation is the highest order derivative present
Highest derivative is $\frac{d^3y}{dx^3}$, so order = 3

Q.1.12 [1 mark]

An integrating factor of the differential equation $\frac{dy}{dx} + y = 3x$ is _

Answer: (c) e^x

Solution:
For linear differential equation $\frac{dy}{dx} + Py = Q$
Integrating factor = $e^{\int P dx} = e^{\int 1 dx} = e^x$

Q.1.13 [1 mark]

$i^7 =$ _

Answer: (b) $-i$

Solution:
$i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1$
$i^7 = i^4 \cdot i^3 = 1 \cdot (-i) = -i$

Q.1.14 [1 mark]

$\arg(1 + i) =$ _

Answer: (c) $\frac{\pi}{4}$

Solution:
$\arg(a+bi) = \tan^{-1}\left(\frac{b}{a}\right)$
$\arg(1+i) = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

Q.2 (A) [6 marks]

Attempt any two

Q.2 (A).1 [3 marks]

If $A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$ **and** $B = \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}$ **then prove that** $(A+B)^T = A^T + B^T$

Solution:
$A + B = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 5 & 3 \end{bmatrix}$

$(A+B)^T = \begin{bmatrix} 6 & 5 \\ 0 & 3 \end{bmatrix}$

$A^T = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$, $B^T = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix}$

$A^T + B^T = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 0 & 3 \end{bmatrix}$

Therefore, $(A+B)^T = A^T + B^T$ ✓ **Proved**

Q.2 (A).2 [3 marks]

If $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ **then show that** $A \cdot A^{-1} = I$

Solution:
First, find A^{-1}:
$|A| = 1(3) - 1(2) = 3 - 2 = 1$

$A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{1}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

Now verify $A \cdot A^{-1} = I$:
$A \cdot A^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1(3)+1(-2) & 1(-1)+1(1) \\ 2(3)+3(-2) & 2(-1)+3(1) \end{bmatrix}$

$= \begin{bmatrix} 3-2 & -1+1 \\ 6-6 & -2+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ ✓ **Proved**

Q.2 (A).3 [3 marks]

Solve the differential equation $xdy + ydx = 0$

Solution:
$xdy + ydx = 0$
$xdy = -ydx$
$\frac{dy}{y} = -\frac{dx}{x}$

Integrating both sides:
$\int \frac{dy}{y} = -\int \frac{dx}{x}$
$\ln|y| = -\ln|x| + c_1$
$\ln|y| + \ln|x| = c_1$
$\ln|xy| = c_1$
$|xy| = e^{c_1} = c$ (where $c = e^{c_1}$ is a constant)

Therefore, $xy = \pm c$ or $xy = k$ where k is an arbitrary constant.

Q.2 (B) [8 marks]

Attempt any two

Q.2 (B).1 [4 marks]

If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ then show that $A^2 - 5A + 7I = 0$

Solution:
First, calculate A^2:
$A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$
$= \begin{bmatrix} 3(3)+1(-1) & 3(1)+1(2) \\ -1(3)+2(-1) & -1(1)+2(2) \end{bmatrix}$
$= \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$

Now calculate $5A$:
$5A = 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}$

And $7I$:
$7I = 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

Now verify $A^2 - 5A + 7I = 0$:
$A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$
$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$ ✓ **Proved**

Q.2 (B).2 [4 marks]

If $A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ **then prove that** $\text{adj } A = A$

Solution:
To find adj A, we need to find the cofactor matrix and then transpose it.

Cofactors:
$C_{11} = (-1)^{1+1} \begin{vmatrix} 0 & 1 \\ 4 & 3 \end{vmatrix} = 0(3) - 1(4) = -4$

$C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} = -(1(3) - 1(4)) = -(3 - 4) = 1$

$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 0 \\ 4 & 4 \end{vmatrix} = 1(4) - 0(4) = 4$

$C_{21} = (-1)^{2+1} \begin{vmatrix} -3 & -3 \\ 4 & 3 \end{vmatrix} = -((-3)(3) - (-3)(4)) = -(-9 + 12) = -3$

$C_{22} = (-1)^{2+2} \begin{vmatrix} -4 & -3 \\ 4 & 3 \end{vmatrix} = (-4)(3) - (-3)(4) = -12 + 12 = 0$

$C_{23} = (-1)^{2+3} \begin{vmatrix} -4 & -3 \\ 4 & 4 \end{vmatrix} = -((-4)(4) - (-3)(4)) = -(-16 + 12) = -(-4) = 4$

$C_{31} = (-1)^{3+1} \begin{vmatrix} -3 & -3 \\ 0 & 1 \end{vmatrix} = (-3)(1) - (-3)(0) = -3$

$C_{32} = (-1)^{3+2} \begin{vmatrix} -4 & -3 \\ 1 & 1 \end{vmatrix} = -((-4)(1) - (-3)(1)) = -(-4 + 3) = -(-1) = 1$

$C_{33} = (-1)^{3+3} \begin{vmatrix} -4 & -3 \\ 1 & 0 \end{vmatrix} = (-4)(0) - (-3)(1) = 0 + 3 = 3$

Cofactor matrix = $\begin{bmatrix} -4 & 1 & 4 \\ -3 & 0 & 4 \\ -3 & 1 & 3 \end{bmatrix}$

$\text{adj } A = (\text{Cofactor matrix})^T = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix} = A$ ✓ **Proved**

Q.2 (B).3 [4 marks]

Solve the following system of linear equations using matrix: $3x + 2y = 5$**,** $2x - y = 1$

Solution:
The system can be written as $AX = B$ where:
$A = \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

Find $|A| = 3(-1) - 2(2) = -3 - 4 = -7$

$$A^{-1} = \frac{1}{-7}\begin{bmatrix} -1 & -2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{2}{7} & -\frac{3}{7} \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{2}{7} & -\frac{3}{7} \end{bmatrix}\begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{7}(5) + \frac{2}{7}(1) \\ \frac{2}{7}(5) - \frac{3}{7}(1) \end{bmatrix} = \begin{bmatrix} \frac{5+2}{7} \\ \frac{10-3}{7} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Therefore, $x = 1, y = 1$

Q.3 (A) [6 marks]

Attempt any two

Q.3 (A).1 [3 marks]

Using definition of differentiation find the derivative of x^5 with respect to x

Solution:
By definition: $\frac{dy}{dx} = \lim_{h\to0} \frac{f(x+h)-f(x)}{h}$

For $f(x) = x^5$:
$\frac{d}{dx}(x^5) = \lim_{h\to0} \frac{(x+h)^5-x^5}{h}$

Using binomial theorem: $(x+h)^5 = x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5$

$\frac{d}{dx}(x^5) = \lim_{h\to0} \frac{x^5+5x^4h+10x^3h^2+10x^2h^3+5xh^4+h^5-x^5}{h}$

$= \lim_{h\to0} \frac{5x^4h+10x^3h^2+10x^2h^3+5xh^4+h^5}{h}$

$= \lim_{h\to0}(5x^4 + 10x^3h + 10x^2h^2 + 5xh^3 + h^4)$

$= 5x^4 + 0 + 0 + 0 + 0 = 5x^4$

Therefore, $\frac{d}{dx}(x^5) = 5x^4$

Q.3 (A).2 [3 marks]

Find $\frac{dy}{dx}$ if $y = \frac{x^2-1}{x^2+1}$

Solution:
Using quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

Here, $u = x^2 - 1$, $v = x^2 + 1$
$\frac{du}{dx} = 2x$, $\frac{dv}{dx} = 2x$

$\frac{dy}{dx} = \frac{(x^2+1)(2x)-(x^2-1)(2x)}{(x^2+1)^2}$

$= \frac{2x(x^2+1)-2x(x^2-1)}{(x^2+1)^2}$

$= \frac{2x[(x^2+1)-(x^2-1)]}{(x^2+1)^2}$

$= \frac{2x[x^2+1-x^2+1]}{(x^2+1)^2}$

$= \frac{2x \cdot 2}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$

Therefore, $\frac{dy}{dx} = \frac{4x}{(x^2+1)^2}$

Q.3 (A).3 [3 marks]

Evaluate the integral $\int \frac{x^2+5x+6}{x^2+2x} dx$

Solution:
First, perform polynomial long division:
$\frac{x^2+5x+6}{x^2+2x} = 1 + \frac{3x+6}{x^2+2x}$

$\int \frac{x^2+5x+6}{x^2+2x} dx = \int \left(1 + \frac{3x+6}{x^2+2x}\right) dx$

$= \int 1 dx + \int \frac{3x+6}{x^2+2x} dx$

$= x + \int \frac{3x+6}{x(x+2)} dx$

For the second integral, use partial fractions:
$\frac{3x+6}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$

$3x + 6 = A(x+2) + Bx$

When $x = 0$: $6 = 2A$, so $A = 3$
When $x = -2$: $-6 + 6 = -2B$, so $B = 0$

Wait, let me recalculate:
When $x = -2$: $3(-2) + 6 = -6 + 6 = 0 = B(-2)$
When $x = 0$: $6 = 2A$, so $A = 3$

Actually: $3x + 6 = 3(x+2)$
So $\frac{3x+6}{x(x+2)} = \frac{3(x+2)}{x(x+2)} = \frac{3}{x}$

$\int \frac{3x+6}{x(x+2)} dx = \int \frac{3}{x} dx = 3 \ln|x| + c_1$

Therefore: $\int \frac{x^2+5x+6}{x^2+2x} dx = x + 3 \ln|x| + c$

Q.3 (B) [8 marks]

Attempt any two

Q.3 (B).1 [4 marks]

If $y = \log(\sec x + \tan x)$ **then find** $\frac{dy}{dx}$

Solution:
$y = \log(\sec x + \tan x)$

$\frac{dy}{dx} = \frac{1}{\sec x + \tan x} \cdot \frac{d}{dx}(\sec x + \tan x)$

$\frac{d}{dx}(\sec x) = \sec x \tan x$
$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{dy}{dx} = \frac{1}{\sec x + \tan x} \cdot (\sec x \tan x + \sec^2 x)$

$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$

$= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} = \sec x$

Therefore, $\frac{dy}{dx} = \sec x$

Q.3 (B).2 [4 marks]

If $y = 2e^{3x} + 3e^{-2x}$ **then prove that** $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$

Solution:
$y = 2e^{3x} + 3e^{-2x}$

First derivative:
$\frac{dy}{dx} = 2(3e^{3x}) + 3(-2e^{-2x}) = 6e^{3x} - 6e^{-2x}$

Second derivative:
$\frac{d^2y}{dx^2} = 6(3e^{3x}) - 6(-2e^{-2x}) = 18e^{3x} + 12e^{-2x}$

Now verify the equation:
$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y$

$= (18e^{3x} + 12e^{-2x}) - (6e^{3x} - 6e^{-2x}) - 6(2e^{3x} + 3e^{-2x})$

$= 18e^{3x} + 12e^{-2x} - 6e^{3x} + 6e^{-2x} - 12e^{3x} - 18e^{-2x}$

$= e^{3x}(18 - 6 - 12) + e^{-2x}(12 + 6 - 18)$

$= e^{3x}(0) + e^{-2x}(0) = 0$ ✓ **Proved**

Q.3 (B).3 [4 marks]

Find the maximum and minimum value of function $f(x) = x^3 - 3x + 11$

Solution:
$f(x) = x^3 - 3x + 11$

First derivative: $f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1)$

For critical points, set $f'(x) = 0$:
$3(x-1)(x+1) = 0$
$x = 1$ or $x = -1$

Second derivative: $f''(x) = 6x$

At $x = 1$: $f''(1) = 6 > 0$ → Local minimum
At $x = -1$: $f''(-1) = -6 < 0$ → Local maximum

Function values:
At $x = 1$: $f(1) = 1^3 - 3(1) + 11 = 1 - 3 + 11 = 9$
At $x = -1$: $f(-1) = (-1)^3 - 3(-1) + 11 = -1 + 3 + 11 = 13$

Therefore:

- **Local maximum value = 13 at** $x = -1$
- **Local minimum value = 9 at** $x = 1$

Q.4 (A) [6 marks]

Attempt any two

Q.4 (A).1 [3 marks]

Evaluate the integral $\int \frac{\cos(\log x)}{x} dx$

Solution:
Let $u = \log x$, then $du = \frac{1}{x} dx$

$\int \frac{\cos(\log x)}{x} dx = \int \cos u \, du = \sin u + c$

Substituting back: $u = \log x$

Therefore, $\int \frac{\cos(\log x)}{x} dx = \sin(\log x) + c$

Q.4 (A).2 [3 marks]

Evaluate the integral $\int x \sin x \, dx$

Solution:
Using integration by parts: $\int u \, dv = uv - \int v \, du$

Let $u = x$ and $dv = \sin x \, dx$
Then $du = dx$ and $v = -\cos x$

$\int x \sin x \, dx = x(-\cos x) - \int(-\cos x) dx$

$= -x \cos x + \int \cos x \, dx$

$= -x \cos x + \sin x + c$

Therefore, $\int x \sin x \, dx = \sin x - x \cos x + c$

Q.4 (A).3 [3 marks]

If $(2x - y) + 2yi = 6 + 4i$ **then find** x **and** y

Solution:
$(2x - y) + 2yi = 6 + 4i$

Comparing real and imaginary parts:
Real part: $2x - y = 6$ ... (1)
Imaginary part: $2y = 4$ ... (2)

From equation (2): $y = 2$

Substituting in equation (1):
$2x - 2 = 6$
$2x = 8$
$x = 4$

Therefore, $x = 4$ **and** $y = 2$

Q.4 (B) [8 marks]

Attempt any two

Q.4 (B).1 [4 marks]

Find the area of the region bounded by the curve $y = x^2$, lines $x = 1$, $x = 2$ and X-axis

Solution:
The required area is given by:
$A = \int_1^2 x^2 \, dx$

$A = \left[\frac{x^3}{3}\right]_1^2$

$= \frac{2^3}{3} - \frac{1^3}{3}$

$= \frac{8}{3} - \frac{1}{3}$

$= \frac{7}{3}$ square units

Therefore, **Area = $\frac{7}{3}$ square units**

Q.4 (B).2 [4 marks]

Evaluate the definite integral $\int_0^{\pi/2} \frac{\sec x}{\sec x + \csc x} dx$

Solution:
Let $I = \int_0^{\pi/2} \frac{\sec x}{\sec x + \csc x} dx$

Using the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$:

$I = \int_0^{\pi/2} \frac{\sec(\pi/2 - x)}{\sec(\pi/2 - x) + \csc(\pi/2 - x)} dx$

Since $\sec(\pi/2 - x) = \csc x$ and $\csc(\pi/2 - x) = \sec x$:

$I = \int_0^{\pi/2} \frac{\csc x}{\csc x + \sec x} dx$

Adding both expressions:
$2I = \int_0^{\pi/2} \frac{\sec x}{\sec x + \csc x} dx + \int_0^{\pi/2} \frac{\csc x}{\sec x + \csc x} dx$

$2I = \int_0^{\pi/2} \frac{\sec x + \csc x}{\sec x + \csc x} dx = \int_0^{\pi/2} 1 \, dx = \frac{\pi}{2}$

Therefore, $I = \frac{\pi}{4}$

Answer: $\int_0^{\pi/2} \frac{\sec x}{\sec x + \csc x} dx = \frac{\pi}{4}$

Q.4 (B).3 [4 marks]

If $\alpha + i\beta = \frac{1}{a+ib}$ then prove that $(\alpha^2 + \beta^2)(a^2 + b^2) = 1$

Solution:
Given: $\alpha + i\beta = \frac{1}{a+ib}$

Rationalizing the right side:
$\alpha + i\beta = \frac{1}{a+ib} \cdot \frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2}$

$\alpha + i\beta = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$

Comparing real and imaginary parts:
$\alpha = \frac{a}{a^2+b^2}$ and $\beta = -\frac{b}{a^2+b^2}$

Now calculating $\alpha^2 + \beta^2$:
$\alpha^2 + \beta^2 = \left(\frac{a}{a^2+b^2}\right)^2 + \left(-\frac{b}{a^2+b^2}\right)^2$

$= \frac{a^2}{(a^2+b^2)^2} + \frac{b^2}{(a^2+b^2)^2}$

$= \frac{a^2+b^2}{(a^2+b^2)^2} = \frac{1}{a^2+b^2}$

Therefore:
$(\alpha^2 + \beta^2)(a^2 + b^2) = \frac{1}{a^2+b^2} \cdot (a^2 + b^2) = 1$ ✓ **Proved**

Q.5 (A) [6 marks]

Attempt any two

Q.5 (A).1 [3 marks]

Find conjugate and modulus of complex number $\frac{2+3i}{3+2i}$

Solution:
First, simplify the complex number by rationalizing:
$\frac{2+3i}{3+2i} = \frac{2+3i}{3+2i} \cdot \frac{3-2i}{3-2i}$

$= \frac{(2+3i)(3-2i)}{(3+2i)(3-2i)}$

$= \frac{6-4i+9i-6i^2}{9-4i^2}$

$= \frac{6+5i-6(-1)}{9-4(-1)}$

$= \frac{6+5i+6}{9+4} = \frac{12+5i}{13}$

So $\frac{2+3i}{3+2i} = \frac{12}{13} + \frac{5}{13}i$

Conjugate: $\overline{\frac{2+3i}{3+2i}} = \frac{12}{13} - \frac{5}{13}i$

Modulus: $\left|\frac{2+3i}{3+2i}\right| = \sqrt{\left(\frac{12}{13}\right)^2 + \left(\frac{5}{13}\right)^2}$

$= \sqrt{\frac{144}{169} + \frac{25}{169}} = \sqrt{\frac{169}{169}} = \sqrt{1} = 1$

Q.5 (A).2 [3 marks]

Simplify: $\frac{(\cos 3\theta + i\sin 3\theta)^{-4}(\cos\theta - i\sin\theta)^{-5}}{(\cos 2\theta - i\sin 2\theta)^7}$

Solution:
Using De Moivre's theorem: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Also, $\cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta)$

$(\cos 3\theta + i\sin 3\theta)^{-4} = \cos(-12\theta) + i\sin(-12\theta)$

$(\cos\theta - i\sin\theta)^{-5} = (\cos(-\theta) + i\sin(-\theta))^{-5} = \cos(5\theta) + i\sin(5\theta)$

$(\cos 2\theta - i\sin 2\theta)^7 = (\cos(-2\theta) + i\sin(-2\theta))^7 = \cos(-14\theta) + i\sin(-14\theta)$

Therefore:
$\frac{(\cos 3\theta + i\sin 3\theta)^{-4}(\cos\theta - i\sin\theta)^{-5}}{(\cos 2\theta - i\sin 2\theta)^7}$

$= \frac{[\cos(-12\theta) + i\sin(-12\theta)][\cos(5\theta) + i\sin(5\theta)]}{\cos(-14\theta) + i\sin(-14\theta)}$

$= \frac{\cos(-12\theta + 5\theta) + i\sin(-12\theta + 5\theta)}{\cos(-14\theta) + i\sin(-14\theta)}$

$= \frac{\cos(-7\theta) + i\sin(-7\theta)}{\cos(-14\theta) + i\sin(-14\theta)}$

$= \cos(-7\theta + 14\theta) + i\sin(-7\theta + 14\theta)$

$= \cos(7\theta) + i\sin(7\theta)$

Q.5 (A).3 [3 marks]

Express Complex number $1 + \sqrt{3}i$ into polar form

Solution:
For complex number $z = a + bi$, polar form is $z = r(\cos\theta + i\sin\theta)$

Here, $a = 1$, $b = \sqrt{3}$

Modulus: $r = |z| = \sqrt{a^2 + b^2} = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1 + 3} = \sqrt{4} = 2$

Argument: $\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

Therefore, the polar form is:
$1 + \sqrt{3}i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

Q.5 (B) [8 marks]

Attempt any two

Q.5 (B).1 [4 marks]

Solve: $\tan y\, dx + \tan x \sec^2 y\, dy = 0$

Solution:
$\tan y\, dx + \tan x \sec^2 y\, dy = 0$

Rearranging: $\tan y\,dx = -\tan x\sec^2 y\,dy$

$\frac{dx}{\tan x} = -\frac{\sec^2 y\,dy}{\tan y}$

$\frac{\cos x}{\sin x}dx = -\frac{dy}{\sin y\cos y}$

$\cot x\,dx = -\frac{dy}{\sin y\cos y}$

Since $\frac{1}{\sin y\cos y} = \frac{2}{2\sin y\cos y} = \frac{2}{\sin 2y}$:

$\cot x\,dx = -\frac{2dy}{\sin 2y}$

Integrating both sides:
$\int\cot x\,dx = -2\int\csc(2y)\,dy$

$\ln|\sin x| = -2\cdot\left(-\frac{1}{2}\ln|\csc(2y)+\cot(2y)|\right) + c$

$\ln|\sin x| = \ln|\csc(2y)+\cot(2y)| + c$

Therefore: $\sin x\cdot[\csc(2y)+\cot(2y)] = k$ where k is a constant.

Q.5 (B).2 [4 marks]

Solve: $x\frac{dy}{dx} - y = x^2$

Solution:
$x\frac{dy}{dx} - y = x^2$

Dividing by x: $\frac{dy}{dx} - \frac{y}{x} = x$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

Here, $P = -\frac{1}{x}$ and $Q = x$

Integrating factor: $I.F. = e^{\int Pdx} = e^{\int -\frac{1}{x}dx} = e^{-\ln|x|} = \frac{1}{x}$

Multiplying the equation by I.F.:
$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = 1$

This can be written as: $\frac{d}{dx}\left(\frac{y}{x}\right) = 1$

Integrating: $\frac{y}{x} = x + c$

Therefore: $y = x^2 + cx$

Q.5 (B).3 [4 marks]

Solve: $\frac{dy}{dx} + \frac{y}{x} = e^x$, $y(0) = 3$

Solution:
This is a linear differential equation: $\frac{dy}{dx} + \frac{y}{x} = e^x$

Here, $P = \frac{1}{x}$ and $Q = e^x$

Integrating factor: $I.F. = e^{\int\frac{1}{x}dx} = e^{\ln|x|} = |x| = x$ (assuming $x > 0$)

Multiplying the equation by I.F.:
$x\frac{dy}{dx} + y = xe^x$

This can be written as: $\frac{d}{dx}(xy) = xe^x$

Integrating both sides:
$xy = \int xe^x dx$

Using integration by parts for $\int xe^x dx$:
Let $u = x$, $dv = e^x dx$
Then $du = dx$, $v = e^x$

$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x = e^x(x-1)$

So: $xy = e^x(x-1) + c$

Therefore: $y = \frac{e^x(x-1)+c}{x}$

Using initial condition $y(0) = 3$:
This presents a problem as we have division by zero. Let me reconsider the approach.

Actually, let's solve this more carefully. The equation $\frac{dy}{dx} + \frac{y}{x} = e^x$ with $y(0) = 3$ has an issue because at $x = 0$, we have division by zero.

For the general solution away from $x = 0$:
$y = \frac{e^x(x-1)+c}{x}$

The initial condition suggests we need to examine the behavior near $x = 0$.

General solution: $y = \frac{e^x(x-1)+c}{x}$ **for** $x \neq 0$

Formula Cheat Sheet

Matrix Operations

- Matrix multiplication: $(AB)_{ij} = \sum_k A_{ik}B_{kj}$
- Inverse of 2×2 matrix: $A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
- Determinant: $|A| = ad - bc$

Differentiation Rules

- Power rule: $\frac{d}{dx}(x^n) = nx^{n-1}$
- Product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
- Quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
- Chain rule: $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Integration Rules

- Power rule: $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ (for $n \neq -1$)
- Integration by parts: $\int u\,dv = uv - \int v\,du$
- Fundamental theorem: $\int_a^b f(x)dx = F(b) - F(a)$

Differential Equations

- Linear first order: $\frac{dy}{dx} + Py = Q$, Solution: $y \cdot I.F. = \int Q \cdot I.F.\ dx$
- Integrating factor: $I.F. = e^{\int Pdx}$
- Variable separable: $\frac{dy}{dx} = f(x)g(y) \rightarrow \frac{dy}{g(y)} = f(x)dx$

Complex Numbers

- Polar form: $z = r(\cos\theta + i\sin\theta)$
- Modulus: $|a + bi| = \sqrt{a^2 + b^2}$
- Argument: $\arg(a + bi) = \tan^{-1}(b/a)$
- De Moivre's theorem: $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

Problem-Solving Strategies

1. **Matrix Problems**: Always check dimensions before multiplication
2. **Differentiation**: Identify which rule applies (product, quotient, chain)
3. **Integration**: Look for substitution opportunities first
4. **Differential Equations**: Identify type (separable vs linear) before solving
5. **Complex Numbers**: Convert to standard form before operations

Common Mistakes to Avoid

1. **Matrix multiplication**: Order matters - $AB \neq BA$ in general
2. **Differentiation**: Don't forget the chain rule for composite functions
3. **Integration**: Always add the constant of integration
4. **Complex numbers**: Be careful with signs when rationalizing

Exam Tips

1. **Time management**: Allocate time based on marks (1 mark = 2-3 minutes)
2. **Show work**: Partial marks are awarded for correct steps
3. **Check units**: Ensure final answers have appropriate units
4. **Verify**: When possible, substitute back to check answers